1 Notes on Laplace circuit analysis

1.1 Background

We previously learned that we can transform from the time domain to the frequency domain under steady-state conditions and thus solve algebraically for the transfer function between the input and output of a circuit. This analysis allowed us to replace indictors and capacitors by their complex impedance, as derivatives in time were replaced by $i\omega$ and integrals in time were replaced by $1/(i\omega)$. The steady-state time dependence is then found by transforming back to the time domain.

The problem is that we often have a signal that turns on at a specific time, which we will take to be $t = 0$ with no loss of generality. Can we calculate the transient behavior with a transform method, as opposed to performing a convolutional integral in the time domain? Let's recall the transform from the time domain to the frequency domain. It is given by:

$$\tilde{V}(\omega) = \int_{-\infty}^{\infty} dt \; v(t) \; e^{-i\omega t}. \tag{1}$$

First, what happens when $\tilde{V}(\omega)$ cannot exist because the integral does not converge at $t = \infty$ and/or $t = -\infty$? For example, suppose $f(t)$=constant or worse yet a polynomial in time? One way to deal is to add an integrating factor; we chose an exponential as this will suppress any polynomial. Thus we add a factor $exp(-a|t|)$ to the integrand, we can take $a \to 0$.

Second, what happens when the system is causal, so that $V(t)$=0 for $t < 0$? Here we take the lower limit as $t = 0$ rather than $t = -\infty$. This is equivalent to multiplying the integrand by a step function, denoted $u(t)$, where

$$u(t) = \begin{cases} 1, & t > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

All of this leads to the Laplace transform:

$$\begin{aligned} V(s) &= \int_{0}^{\infty} dt \; v(t) \; e^{-at} e^{-i\omega t} \\ &= \int_{0}^{\infty} dt \; v(t) \; e^{-st} \end{aligned} \tag{3}$$

where $s = a + i\omega$ and $v(t)$ is understood as $v(t)u(t)$. Keep in mind that the units of $V(s)$ are Volts $\times$ time. The inverse transform is a bit more involved, but we will show how this can be readily done for any of the functions that arise in linear circuit analysis. We have

$$v(t)u(t) = \frac{1}{2\pi i} \int_C ds \; V(s) \; e^{st}, \tag{4}$$

which is a contour integral in the complex s-plane.

All we need to know about, at least to start analyzing the kind of circuits familiar to the class, are two rules

$$\frac{dv(t)}{dt} \Rightarrow \int_0^\infty dt\, \frac{dv(t)}{dt}\, e^{-st} = \int_0^\infty dt\, s\, v(t)\, e^{-st} + v(t)\, e^{-st}|_0^\infty = sV(s) - v(0) \tag{5}$$

$$\int_0^t dx\, v(x) \Rightarrow \int_0^\infty dt \int_0^t dx\, v(x)\, e^{-st} = \cdots = \frac{1}{s}V(s) \tag{6}$$

and three transforms

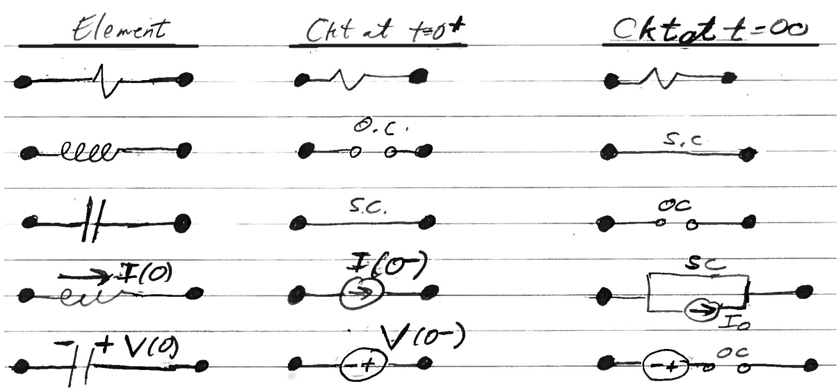
$$1 \Rightarrow \int_0^\infty dt\, e^{-st} = \frac{1}{s} \tag{7}$$

$$e^{-at} \Rightarrow \int_0^\infty dt\, e^{-at}\, e^{-st} = \frac{1}{s+a} \tag{8}$$

$$sin\omega t \Rightarrow \int_0^\infty dt\, sin\omega t\, e^{-st} = \frac{\omega}{s^2+\omega^2} \tag{9}$$

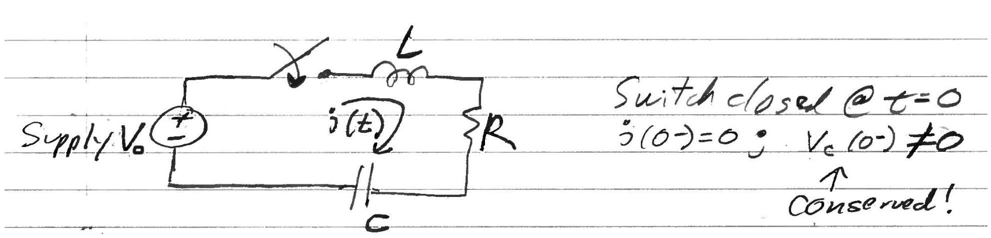
$$cos\omega t \Rightarrow \frac{1}{\omega}\frac{d\, sin(\omega t)}{dt} = \frac{s}{\omega}\frac{\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} \tag{10}$$

Note that the derivative transform includes initial conditions, as shown in the table:



1.2 Application with step-to-constant input

Let's apply our new knowledge to a circuit that has a switch that closes at time $t = 0$. Thus the current at $t = 0^+$ equals the current at $t = 0^-$, which is $I = 0$ since the current through an inductor cannot change instantaneously. The initial voltage across the capacitor however, may not be zero. This $V_C(0^+) = V_C(0^-)$ since the voltage across the capacitor cannot change instantaneously,



The equations are:

$$- v_o \; + \; L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t dt' \; i(t') + V_C(0^-) = 0. \tag{11}$$

Transforming, we get

$$\frac{-v_o}{s} \; + \; LsI(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) + \frac{V_C(0^-)}{s} = 0. \tag{12}$$

We multiply through bys/L terms to get

$$-\frac{v_o}{L} \; + \; s^2 I(s) + \frac{R}{L}sI(s) + \frac{I(s)}{LC} + \frac{V_C(0^-)}{L} = 0 \tag{13}$$

so that

$$I(s) = \left(\frac{v_o \; - \; V_C(0^-)}{L}\right)\frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}}. \tag{14}$$

We let $V_C(0^-)$ = 0 simply to minimize the algebra in the following mathematics. Thus:

$$I(s) = \frac{v_o}{L}\,\frac{1}{s^2 + 2ks + \omega_o^2} \tag{15}$$

where $k = \frac{R}{2L}$ is a decay rate and $\omega_o = \frac{1}{\sqrt{LC}}$ is a resonant frequency. The first thing we need to do is factor the denominator. We have

$$\text{roots} = -k \pm i\sqrt{\omega_o^2 - k^2} \tag{16}$$

Thus

$$I(s) = \frac{v_o}{L}\frac{1}{(s-a)(s-a^*)} \tag{17}$$

with

$$a = -k + i\sqrt{\omega_o^2 - k^2} \tag{18}$$

and thus

$$i(t) = \frac{v_o}{L}\frac{1}{2\pi i}\int_C ds\frac{e^{st}}{(s-a)(s-a^*)}. \tag{19}$$

In order to solve this we need a refresher on the residue theorem

"Blitz refresher on Cauchy's Residue Theorem"

Integrals in the complex plane, of the form used in linear circuit analysis, may be evaluated by

$$\int_C dsF(s) = 2\pi i \; \Sigma \; \text{Residues}.$$

When

$$\begin{aligned} F(s) &= \frac{\text{Any regular function}}{\text{Polynomial function with simple zeros}} \\ &\equiv \frac{q(s)}{p(s)} \end{aligned}$$

the residue at each zero of $p(s)$, or pole of $F(s)$, is given by the expression

$$\text{Residue} = \frac{q(s)}{\frac{\partial p(s)}{\partial s}}|_{s=s_{\text{pole}}}.$$

For example, with $q(s) = r(s)e^{st}$ and $p(s) = (s-a)(s-b)\cdots(s-y)(s-z)$, we have

$$\begin{aligned}\int_C dsF(s) &= \int_C ds\frac{r(s)e^{st}}{(s-a)(s-b)(s-c)\cdots(s-y)(s-z)}\\ &= 2\pi i\left[\frac{r(s)e^{st}}{(s-b)\cdots(s-y)(s-z)}|_{s=a} + \cdots + \frac{r(s)e^{st}}{(s-a)(s-b)\cdots(s-y)}|_{s=z}\right]\\ &= 2\pi i\left[\frac{r(a)e^{at}}{(a-b)\cdots(a-y)(a-z)} + \cdots + \frac{r(z)e^{zt}}{(z-a)(z-b)\cdots(z-y)}\right].\end{aligned}$$

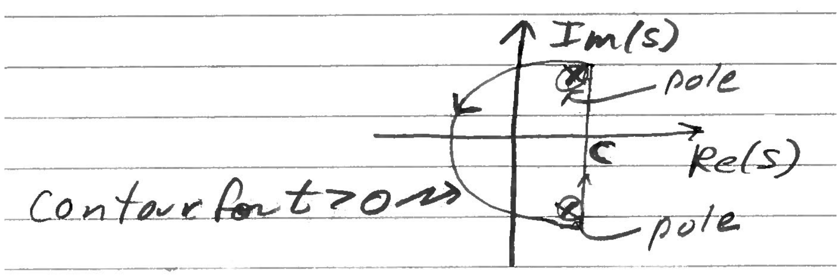
Note that complex poles always appear as conjugate pairs. Thus, for example, with

$$F(s) = \frac{e^{st}}{(s-a)(s-a^*)}$$

we find

$$\begin{aligned}f(t) &= \frac{1}{2\pi i}\int_C dsF(s)\\ &= \frac{1}{2\pi i}\int_C ds\frac{e^{st}}{(s-a)(s-a^*)}\\ &= \frac{1}{2\pi i}\,2\pi i\left[\frac{e^{at}}{(a-a^*)} + \frac{e^{a^*t}}{(a^*-a)}\right].\\ &= e^{\mathrm{Re}[a]t}\left[\frac{e^{i\mathrm{Im}[a]t} - e^{-i\mathrm{Im}[a]t}}{2i\mathrm{Im}[a]}\right]\\ &= \frac{e^{\mathrm{Re}[a]t}}{\mathrm{Im}[a]}\,sin(\mathrm{Im}[a]t)\end{aligned}$$

which is just the form of our solution for the prior circuit application.



The Cauchy residue theorem for the inverse transform thus yields:

$$\begin{aligned}i(t) &= \frac{v_o}{L}\frac{e^{\mathrm{Re}[a]t}}{\mathrm{Im}[a]}\,sin(\mathrm{Im}[a]t) \qquad (20)\\ &= \frac{v_o}{L}\frac{e^{-kt}}{\sqrt{\omega_o^2-k^2}}sin\left(\sqrt{\omega_o^2-k^2}\,t\right).\end{aligned}$$

Note that the shift in the natural frequency, from ω_o to $\omega_o\left[1-\frac{1}{2}(\frac{k}{\omega_o})^2+\cdots\right]$, is quite clear. When the loss is high, i.e., $k > \omega_o$, the sine term becomes a hyperbolic sine and the current just rises and decays exponentially. For the special case of $k = \omega_o$, so called 'critical damping',

$$i(t) = \frac{2v_o}{R}\,kt\,e^{-kt}. \qquad (21)$$

Note also that the current at very short times is limited by the highest impedance, which is the inductance. In particular

$$i(t) \quad \overrightarrow{t \to 0} \quad \frac{v_o}{L}t. \tag{22}$$

1.3 Application with step-to-sinusoid (tone) input

Let's now move to a more interesting dynamics and replace the source with a cosine that turns on at $t = 0$, that is

$$v_o(t) = v_o cos(\omega t) \tag{23}$$

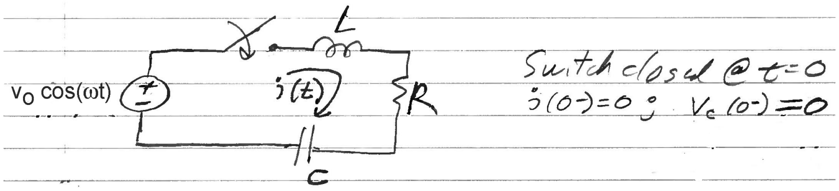
so that

$$- v_o cos(\omega t) \ + \ L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t dt' \ i(t') = 0 \tag{24}$$

where, for simplicity, we take the initial voltage on the capacitor to be zero. Transforming, we get

$$\begin{aligned} I(s) &= i_o\frac{k \ s^2}{(s-a)(s-a^*)(s^2-\omega^2)} \\ &= i_o\frac{k \ s^2}{(s-a)(s-a^*)(s-i\omega)(s+i\omega)}. \end{aligned} \tag{25}$$

where we use the same abbreviations as above. We choose to use $cos(\omega t)u(t)$ as this reverts to $u(t)$ as $\omega \to 0$.



The circuit will respond at both the driven frequency and at the natural frequency. We have, from the residue theorem, four terms that we will evaluate in pairs, i.e.

$$i(t) = i_o[f_1(t) + f_2(t)] \tag{26}$$

where

$$f_1(t) = \frac{k \ s^2 \ e^{st}}{(s-a^*)(s^2+\omega^2)}|_{s=a} + \frac{k \ s^2 \ e^{st}}{(s-a)(s^2+\omega^2)}|_{s=a^*} \tag{27}$$

and

$$f_2(t) = \frac{k \ s^2 \ e^{st}}{(s-a)(s-a^*)(s+i\omega)}|_{s=i\omega} + \frac{k \ s^2 \ e^{st}}{(s-a)(s-a^*)(s-i\omega)}|_{s=-i\omega}. \tag{28}$$

Before we solder on with $f_1(t)$, let's calculate some of the algebraic terms that we will need, i.e.,

$$a - a^* = i2\sqrt{\omega_o^2 - k^2}, \tag{29}$$

$$aa^* = \omega_o^2, \tag{30}$$

$$a^2 = 2k^2 - \omega_o^2 + i2\sqrt{\omega_o^2 - k^2}. \tag{31}$$

and

$$a^{*2} = 2k^2 - \omega_o^2 - i2\sqrt{\omega_o^2 - k^2}. \tag{32}$$

$$f_1(t) = \frac{ke^{-kt}}{\sqrt{\omega_o^2-k^2}}\frac{1}{2i}\left(\frac{2k^2-\omega_o^2+i2k\sqrt{\omega_o^2-k^2}}{2k^2+\omega^2-\omega_o^2+i2k\sqrt{\omega_o^2-k^2}}e^{i\sqrt{\omega_o^2-k^2}\ t} - \frac{2k^2-\omega_o^2-i2k\sqrt{\omega_o^2-k^2}}{2k^2+\omega^2-\omega_o^2-i2k\sqrt{\omega_o^2-k^2}}e^{-i\sqrt{\omega_o^2-k^2}\ t}\right). \tag{33}$$

We rationalize the denominator, noting that

$$\left(2k^2+\omega^2-\omega_o^2+i2k\sqrt{\omega_o^2-k^2}\right)\left(2k^2+\omega^2-\omega_o^2-i2k\sqrt{\omega_o^2-k^2}\right) = (\omega^2-\omega_o^2)^2+(2k\omega)^2, \tag{34}$$

$$\left(2k^2-\omega_o^2+i2k\sqrt{\omega_o^2-k^2}\right)\left(2k^2+\omega^2-\omega_o^2-i2k\sqrt{\omega_o^2-k^2}\right) = \omega_o^4-\omega^2(\omega_o^2-2k^2)+i2k\omega^2\sqrt{\omega_o^2-k^2} \tag{35}$$

and

$$\left(2k^2-\omega_o^2-i2k\sqrt{\omega_o^2-k^2}\right)\left(2k^2+\omega^2-\omega_o^2+i2k\sqrt{\omega_o^2-k^2}\right) = \omega_o^4-\omega^2(\omega_o^2-2k^2)-i2k\omega^2\sqrt{\omega_o^2-k^2} \tag{36}$$

so that

$$\begin{aligned}
f_1(t) &= \frac{ke^{-kt}}{\sqrt{\omega_o^2-k^2}[(\omega^2-\omega_o^2)^2+(2k\omega)^2]} \qquad (37)\\
&\times \left(\left[\omega_o^4-\omega^2(\omega_o^2-2k^2)\right]\frac{e^{i\sqrt{\omega_o^2-k^2}t}-e^{-i\sqrt{\omega_o^2-k^2}t}}{2i} + 2k\omega^2\sqrt{\omega_o^2-k^2}\frac{e^{i\sqrt{\omega_o^2-k^2}t}+e^{-i\sqrt{\omega_o^2-k^2}t}}{2}\right)\\
&= \frac{ke^{-kt}}{\sqrt{\omega_o^2-k^2}[(\omega^2-\omega_o^2)^2+(2k\omega)^2]}\\
&\times \left(\left[\omega_o^4-\omega^2(\omega_o^2-2k^2)\right] sin(\sqrt{\omega_o^2-k^2}\ t) + 2k\omega^2\sqrt{\omega_o^2-k^2}\ cos(\sqrt{\omega_o^2-k^2}\ t)\right)\\
&= \frac{\omega_o^2 ke^{-kt}}{\sqrt{(\omega_o^2-k^2)[(\omega^2-\omega_o^2)^2+(2k\omega)^2]}}\\
&\times \left(\frac{\omega_o^4-\omega^2(\omega_o^2-2k^2)}{\omega_o^2\sqrt{(\omega^2-\omega_o^2)^2+(2k\omega)^2}} sin(\sqrt{\omega_o^2-k^2}\ t) + \frac{2k\omega^2\sqrt{\omega_o^2-k^2}}{\omega_o^2\sqrt{(\omega^2-\omega_o^2)^2+(2k\omega)^2}}\ cos(\sqrt{\omega_o^2-k^2}\ t)\right).
\end{aligned}$$

The weighting factors for the sine and cosine terms satisfy the right triangle rule

$$\left[\omega_o^4-\omega^2(\omega_o^2-2k^2)\right]^2 + \left[2k\omega^2\sqrt{\omega_o^2-k^2}\right]^2 = \left[\omega_o^2\sqrt{(\omega^2-\omega_o^2)^2+(2k\omega)^2}\right]^2 \tag{38}$$

so that with the definition

$$\phi_1(\omega) = atan\left(\frac{2k\omega^2\sqrt{\omega_o^2-k^2}}{\omega_o^4-\omega^2(\omega_o^2-2k^2)}\right) \tag{39}$$

we have

$$\begin{aligned}
f_1(t) &= \frac{\omega_o^2 ke^{-kt}}{\sqrt{(\omega_o^2-k^2)[(\omega^2-\omega_o^2)^2+(2k\omega)^2]}} \qquad (40)\\
&\times \left(cos[\phi_1(\omega)]\ sin(\sqrt{\omega_o^2-k^2}\ t)\ +\ sin[\phi_1(\omega)]\ cos(\sqrt{\omega_o^2-k^2}\ t)\right)\\
&= \frac{\omega_o^2}{\sqrt{(\omega^2-\omega_o^2)^2+(2k\omega)^2}}\frac{ke^{-kt}}{\sqrt{(\omega_o^2-k^2)}}\ sin\left(\sqrt{\omega_o^2-k^2}\ t+\phi_1(\omega)\right).
\end{aligned}$$

The first term is maximized for the choice of drive frequency $\omega = \sqrt{\omega_o^2-2k^2}$, which is slightly lower than the natural response frequency of $\omega = \sqrt{\omega_o^2-k^2}$. Lastly, and as a sanity check, in the limit of $\omega \to 0$, we recover the result for the response to a step input, i.e.

$$f_1(t) \ \overrightarrow{\omega \to 0}\ \frac{ke^{-kt}}{\sqrt{\omega_o^2-k^2}}\ sin\left(\sqrt{\omega_o^2-k^2}\ t\right). \tag{41}$$

Let's now move on to the driven term $f_2(t)$. We first note the evaluations:

$$(s-a)(s-a^*)|_{s=i\omega} = (s^2+2ks+\omega_o^2)|_{s=i\omega} = (\omega_o^2-\omega^2)+i2k\omega, \tag{42}$$

$$(s-a)(s-a^*)|_{s=-i\omega} = (s^2+2ks+\omega_o^2)|_{s=-i\omega} = (\omega_o^2-\omega^2)-i2k\omega. \tag{43}$$

and

$$[(\omega_o^2-\omega^2)+i2k\omega][(\omega_o^2-\omega^2)-i2k\omega] = (\omega_o^2-\omega^2)^2+(2k\omega)^2 \tag{44}$$

Then

$$\begin{aligned}
f_2(t) &= \frac{k\ (i\omega)^2\ e^{i\omega t}}{[(\omega_o^2-\omega^2)+i2k\omega](i2\omega)} + \frac{k\ (-i\omega)^2\ e^{-i\omega t}}{[(\omega_o^2-\omega^2)-i2k\omega](-i2\omega)} \\
&= \frac{-k\omega}{2i}\left(\frac{e^{i\omega t}[(\omega_o^2-\omega^2)-i2k\omega]\ -\ e^{-i\omega t}[(\omega_o^2-\omega^2)+i2k\omega]}{(\omega_o^2-\omega^2)^2+(2k\omega)^2}\right) \\
&= \frac{k\omega}{(\omega^2-\omega_o^2)^2+(2k\omega)^2}\left(2k\omega\frac{e^{i\omega t}+e^{-i\omega t}}{2}\ +(\omega^2-\omega_o^2)\frac{e^{i\omega t}+e^{-i\omega t}}{2i}\right) \\
&= \frac{k\omega}{(\omega^2-\omega_o^2)^2+(2k\omega)^2}\left((\omega^2-\omega_o^2)\ sin(\omega t)\ +\ 2k\omega\ cos(\omega t)\right) \\
&= \frac{k\omega}{\sqrt{(\omega^2-\omega_o^2)^2+(2k\omega)^2}}\left(\frac{\omega^2-\omega_o^2}{\sqrt{(\omega^2-\omega_o^2)^2+(2k\omega)^2}}\ sin(\omega t)\ +\ \frac{2k\omega}{\sqrt{(\omega^2-\omega_o^2)^2+(2k\omega)^2}}\ cos(\omega t)\right)
\end{aligned} \tag{45}$$

The weighting factors for the sine and cosine terms satisfy the right triangle rule, so

$$\begin{aligned}
f_2(t) &= \frac{k\omega}{\sqrt{(\omega^2-\omega_o^2)^2+(2k\omega)^2}}\left(sin[\phi_2(\omega)]sin(\omega t)+\cos[\phi_2(\omega)]cos(\omega t)\right) \\
&= \frac{k\omega}{\sqrt{(\omega^2-\omega_o^2)^2+(2k\omega)^2}}cos\left(\omega t-\phi_2(\omega)\right)
\end{aligned} \tag{46}$$

where

$$\phi_2(\omega) = atan\left(\frac{\omega^2-\omega_o^2}{2k\omega}\right) \tag{47}$$

$$\begin{aligned}
i(t) &= \frac{v_o}{2kL}\left(f_1(t)+f_2(t)\right) \\
&= \frac{v_o}{2\omega_o L}\frac{\omega_o^2}{\sqrt{(\omega^2-\omega_o^2)^2+(2k\omega)^2}}\left(\frac{\omega_o e^{-kt}}{\sqrt{(\omega_o^2-k^2)}}\ sin\left(\sqrt{\omega_o^2-k^2}\ t+\phi_1(\omega)\right)+\frac{\omega}{\omega_o}cos\left(\omega t-\phi_2(\omega)\right)\right).
\end{aligned} \tag{48}$$

At $t=0^+$, the amplitude of the response is

$$i(0^+) = \frac{v_o}{R}\ \frac{(2k\omega)^2}{(\omega^2-\omega_o^2)^2+(2k\omega)^2} \tag{49}$$