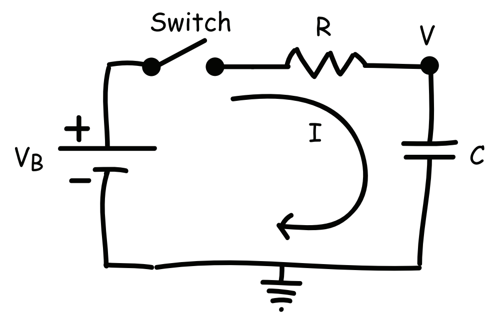
Review of RC Circuits

This handout covers current and voltage changes in a single-loop circuit with a resistor and a capacitor.



For the circuit above the initial charge of the capacitor is zero and the switch is closed at time $t = 0^+$. We have:

$$0 = -V_B + IR + \frac{Q}{C} .$$

Recall that the current is just the change in charge with response to time, i.e., $I = dQ/dt$. This charge must end up on the capacitor plates. So:

$$0 = -V_B + \frac{dQ}{dt}R + \frac{Q}{C} .$$

A little rearranging to place each term in the form of a current gives:

$$\frac{V_B}{R} = \frac{dQ}{dt} + \frac{Q}{\tau} .$$

The solution of this equation - recall that $\int_a^b \frac{dx}{x} = \log_e(b) - \log_e(a)$ and - try to solve it yourself – is:

$$Q(t) = CV_B\left(1 - e^{-t/\tau}\right) .$$

Thus $I = dQ/dt$ is:

$$I(t) = \frac{V_B}{R}e^{-t/\tau}$$

and $I(t = 0^+) = V_B/R$ and $I(t \rightarrow \infty) = 0$. With $V = Q/C$, the voltage across the capacitor is:

$$V(t) = V_B\left(1 - e^{-t/\tau}\right).$$

and $V(t = 0^+) = 0$ and $V(t \rightarrow \infty) = V_B$.

A final point concerns the energy stored in the capacitor. The instantaneous power is just:

$$P(t) = I(t)\,V(t)$$

$$= \frac{V_B^2}{R} e^{-t/\tau}\left(1 - e^{-t/\tau}\right)$$

$$= \frac{V_B^2}{R}\left(e^{-t/\tau} - e^{-2t/\tau}\right)$$

The energy stored after the capacitor charges is the integral of the power, i.e.,

$$E = \int_0^\infty dt\, P(t)$$

This becomes - recall that $\int_a^b dx\, e^{\lambda x} = \frac{e^{\lambda b} - e^{\lambda a}}{\lambda}$ and try to solve it yourself! –

$$E = \frac{V_B^2}{R}\int_0^\infty dt\left(e^{-t/\tau} - e^{-2t/\tau}\right)$$

$$= \frac{V_B^2}{R}\left(\frac{e^{-\infty} - e^0}{-1/\tau} - \frac{e^{-\infty} - e^0}{-2/\tau}\right)$$

$$= \frac{V_B^2}{R}\left(\frac{0-1}{-1/RC} - \frac{0-1}{-2/RC}\right)$$

$$= \frac{V_B^2}{R}\left(RC - \frac{RC}{2}\right)$$

$$= \frac{1}{2} C\, V_B^2$$

which is the expected result for energy in a capacitor.

Bottom Line: A resistor/capacitor paircharge with a characteristic time ($\tau \equiv$ time constant) that is given by the product of the resistance and membrane ($\tau = RC$).

In one time–constant, the capacitor (or membrane) reaches $(1 - e^{-1}) \times 100\ \% = 63\ \%$ of its final charge (or voltage).

As a side issue, for isopotential biological membranes $RC = \rho\frac{d}{A}\frac{\kappa}{4\pi k_e}\frac{A}{d} = \frac{\kappa\rho}{4\pi k_e}$, which is indepenedent of the geometry.