

1 Notes on time domain circuit analysis

1.1 Background

The most common first-order differential equation in circuit analysis, such as for a RC high-pass or low-pass filter, has constant coefficients and is of the form

$$\tau \frac{dV(t)}{dt} + V(t) = F(t). \quad (1.1)$$

The full solution is the solution to the homogeneous equation, denoted $V_H(t)$ plus the solution to the inhomogeneous solution, denoted $V_I(t)$. The homogeneous equation is the part with $F(t) = 0$, i.e., the equation for the decay from $V(0)$ for which the homogeneous solution is, to within a constant,

$$V_H(t) = V(0^+) \Phi(t) \quad (1.2)$$

with

$$\Phi(t) = e^{-t/\tau}. \quad (1.3)$$

The inhomogeneous solution satisfies the full equation

$$\tau \frac{dV_I(t)}{dt} + V_I(t) = F(t) \quad (1.4)$$

and we take as an ansatz that the inhomogeneous solution is proportion to the homogeneous solution times a yet to be determined function of time $\alpha(t)$, i.e.,

$$\begin{aligned} V_I(t) &= \Phi(t) \alpha(t) \\ &= e^{-t/\tau} \alpha(t) \end{aligned} \quad (1.5)$$

and

$$\tau \frac{dV_I(t)}{dt} = \tau e^{-t/\tau} \frac{d\alpha(t)}{dt} - \tau e^{-t/\tau} \frac{1}{\tau} \alpha(t). \quad (1.6)$$

If we plug these back into the original equation, we get

$$\tau e^{-t/\tau} \frac{d\alpha(t)}{dt} - e^{-t/\tau} \alpha(t) + e^{-t/\tau} \alpha(t) = F(t) \quad (1.7)$$

or

$$\frac{d\alpha(t)}{dt} = \frac{1}{\tau} e^{t/\tau} F(t) \quad (1.8)$$

or

$$\alpha(t) = \int_0^t \frac{dt'}{\tau} e^{t'/\tau} F(t') + \text{Constant} \quad (1.9)$$

and, substituting back for $\alpha(t)$ gives

$$\begin{aligned} V_I(t) &= e^{-t/\tau} \int_0^t \frac{dt'}{\tau} e^{t'/\tau} F(t') + \text{Constant} \\ &= \int_0^t \frac{dt'}{\tau} e^{-(t-t')/\tau} F(t') + \text{Constant}. \end{aligned} \quad (1.10)$$

Thus

$$\begin{aligned} V(t) &= V_I(t) + V_H(t) \\ &= \int_0^t \frac{dt'}{\tau} e^{-(t-t')/\tau} F(t') + V(0^+)e^{-t/\tau} \end{aligned} \quad (1.11)$$

The integral for the driven response is call the "convolution" integral. The above relation is a special case, for constant coefficients, between the input and the driven response for a linear system. In general, the response of such a linear system to a pulse ("delta function") is given by $\Phi(t)$ and

$$V_I(t) = \int_0^t \frac{dt'}{\tau} \Phi(t-t') F(t'). \quad (1.12)$$

1.2 Example of step input

Here we have a signal that is $F(t) = 0$ for $t < 0$ and $F(t) = V_0$ for $t \geq 0$.

$$\begin{aligned} V(t) &= \int_0^t \frac{dt'}{\tau} e^{-(t-t')/\tau} V_0 + V(0^+)e^{-t/\tau} \\ &= V_0 (1 - e^{-t/\tau}) + V(0^+)e^{-t/\tau} \end{aligned} \quad (1.13)$$

1.3 Example of sine input

Here we have a signal that is $F(t) = 0$ for $t < 0$ and $F(t) = V_0 \sin(\omega_0 t)$ for $t \geq 0$. For focus only on the driven part and take $V(0^-) = 0$.

$$\begin{aligned} V(t) &= \int_0^t \frac{dt'}{\tau} e^{-(t-t')/\tau} V_0 \sin(\omega_0 t') \\ &= V_0 e^{-t/\tau} \int_0^t \frac{dt'}{\tau} e^{t'/\tau} \frac{e^{i\omega_0 t'} - e^{-i\omega_0 t'}}{2i} \\ &= V_0 \frac{e^{-t/\tau}}{2i} \int_0^{t/\tau} dx e^x (e^{i\omega_0 \tau x} - e^{-i\omega_0 \tau x}) \end{aligned} \quad (1.14)$$

$$\begin{aligned}
&= V_0 \frac{e^{-t/\tau}}{2i} \int_0^{t/\tau} dx \left(e^{(1+i\omega_0\tau)x} - e^{(1-i\omega_0\tau)x} \right) \\
&= V_0 \frac{e^{-t/\tau}}{2i} \left(\frac{e^{t/\tau} e^{i\omega_0 t} - 1}{1 + i\omega_0\tau} - \frac{e^{t/\tau} e^{-i\omega_0 t} - 1}{1 - i\omega_0\tau} \right) \\
&= V_0 \frac{1}{2i} \left[\left(\frac{e^{i\omega_0 t}}{1 + i\omega_0\tau} - \frac{e^{-i\omega_0 t}}{1 - i\omega_0\tau} \right) - e^{-t/\tau} \left(\frac{1}{1 + i\omega_0\tau} - \frac{1}{1 - i\omega_0\tau} \right) \right] \\
&= V_0 \frac{1}{2i} \left[\frac{(1 - i\omega_0\tau)e^{i\omega_0 t} - (1 + i\omega_0\tau)e^{-i\omega_0 t} + 2i\omega_0\tau e^{-t/\tau}}{1 + (\omega_0\tau)^2} \right] \\
&= V_0 \frac{1}{1 + (\omega_0\tau)^2} \left[\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} - \omega_0\tau \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} + \omega_0\tau e^{-t/\tau} \right] \\
&= V_0 \left[\frac{1}{1 + (\omega_0\tau)^2} \sin(\omega_0 t) - \frac{\omega_0\tau}{1 + (\omega_0\tau)^2} \cos(\omega_0 t) + \frac{\omega_0\tau}{1 + (\omega_0\tau)^2} e^{-t/\tau} \right]
\end{aligned}$$

The sine term represents a faithful transmission of the input and the cosine term represents a phase shifted version, while the exponential represents the transient from turning the signal on at $t=0$. With a bit more algebra, and recalling that $\sin(a - b) = \sin(a) \cos(b) - \cos(a) \sin(b)$, we can simplify this.

$$\begin{aligned}
V(t) &= \frac{V_0}{\sqrt{1 + (\omega_0\tau)^2}} \left[\frac{1}{\sqrt{1 + (\omega_0\tau)^2}} \sin(\omega_0 t) - \frac{\omega_0\tau}{\sqrt{1 + (\omega_0\tau)^2}} \cos(\omega_0 t) + \frac{\omega_0\tau}{\sqrt{1 + (\omega_0\tau)^2}} e^{-t/\tau} \right] \\
&= \frac{V_0}{\sqrt{1 + (\omega_0\tau)^2}} \left[\sin[\omega_0 t - \text{atan}(\omega_0\tau)] + \frac{\omega_0\tau}{\sqrt{1 + (\omega_0\tau)^2}} e^{-t/\tau} \right] \tag{1.15}
\end{aligned}$$

At short times, i.e., $t \ll \tau$,

$$V(t) \approx \frac{\omega_0\tau}{1 + (\omega_0\tau)^2} \frac{t}{\tau} \tag{1.16}$$

and in steady state, i.e., $t \gg \tau$,

$$V(t) = \frac{V_0}{\sqrt{1 + (\omega_0\tau)^2}} \sin[\omega_0 t - \text{atan}(\omega_0\tau)] \tag{1.17}$$