6 Week 6e - Time domain circuit analysis.

6.1 Background

The most common first-order differential equation in circuit analysis, such as for a RC high-pass or low-pass filter, has constant coefficients and is of the form

$$\tau \frac{dV(t)}{dt} \; + \; V(t) \; = \; F(t). \tag{6.6}$$

The full solution is the solution to the homogeneous equation, denoted $V_H(t)$ plus the solution to the inhomogeneous solution, denoted $V_I(t)$. The homogeneous equation is the part with $F(t) = 0$, i.e., the equation for the decay from $V(0)$ for which the homogeneous solution is, to within a constant,

$$V_H(t) \; = \; V(0^+)h(t) \tag{6.7}$$

with

$$h(t) = e^{-t/\tau}. \tag{6.8}$$

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The inhomogeneous solution satisfies the full equation

$$\tau \frac{dV_I(t)}{dt} \; + \; V_I(t) \; = \; F(t) \tag{6.9}$$

and we take as an ansatz that the inhomogeneous solution is proportion to the homogeneous solution times a yet to be determined function of time $u(t)$, i.e.,

$$\begin{aligned} V_I(t) \quad &= \quad h(t)u(t) \\ &= \quad e^{-t/\tau}u(t) \end{aligned} \tag{6.10}$$

and

$$\tau \frac{dV_I(t)}{dt} \; = \; \tau e^{-t/\tau}\frac{du(t)}{dt} - \tau e^{-t/\tau}\frac{1}{\tau}u(t). \tag{6.11}$$

If we plug these back into the original equation, we get

$$\tau e^{-t/\tau}\frac{du(t)}{dt} - e^{-t/\tau}u(t) + e^{-t/\tau}u(t) = F(t) \tag{6.12}$$

or

$$\frac{du(t)}{dt} \; = \; \frac{1}{\tau}e^{t/\tau}F(t) \tag{6.13}$$

or

$$u(t) \; = \int_0^t \frac{dt'}{\tau} e^{t'/\tau} \; F(t') \; + \; \text{Constant} \tag{6.14}$$

and, substituting back for $u(t)$ gives

$$\begin{aligned} V_I(t) &= e^{-t/\tau} \int_0^t \frac{dt'}{\tau} e^{t'/\tau}\ F(t')\ +\ \text{Constant} \\ &= \int_0^t \frac{dt'}{\tau} e^{-(t-t')/\tau}\ F(t')\ +\ \text{Constant}. \end{aligned} \tag{6.15}$$

Thus

$$\begin{aligned} V(t) &= V_I(t)\ +\ V_H(t) \\ &= \int_0^t \frac{dt'}{\tau}\ e^{-(t-t')/\tau}\ F(t')\ +\ V(0^+)e^{-t/\tau} \end{aligned} \tag{6.16}$$

The integral for the driven response is call the "convolution" integral. The above relation is a special case, for constant coefficients, between the input and the driven response for a linear system.

In general, the response of such a linear system to a pulse ("delta function") is given by $h(t)$, i.e.,

$$h(t)\ =\ \int_0^t \frac{dt'}{\tau}\ h(t-t')\ \delta(t'). \tag{6.17}$$

so that

$$V_I(t)\ =\ \int_0^t \frac{dt'}{\tau}\ h(t-t')\ F(t'). \tag{6.18}$$

One can view $F(t')$ as the distribution of arrival times of a train of individual pulses, so that

$$V_I(t)\ =\ \sum_0^N\ h(t-n\Delta t')\ A(n\Delta t')\delta(n\Delta t') \tag{6.19}$$

where $A(n\Delta t')$ is the amplitude of the pulse at time $n\Delta t'$ and in the limit of $\Delta t' \to 0$, $A(n\Delta t')\delta(n\Delta t') \to F(t')$. This is the essence of linear response theory; that the output can we written in terms of the input and the response of the system to pulsed inputs. Much as we have presented this for time-depence through circuits, the concept of linear response is general across space ad time.

6.2 Example of step input

Here we have a signal that is $F(t) = 0$ for $t < 0$ and $F(t) = V_0$ for $t \geq 0$.

$$\begin{aligned} V(t) &= \int_0^t \frac{dt'}{\tau}\ e^{-(t-t')/\tau}\ V_0\ +\ V(0^+)e^{-t/\tau} \\ &= V_0\ (1\ -\ e^{-t/\tau})\ +\ V(0^+)e^{-t/\tau} \end{aligned} \tag{6.20}$$

6.3 Example of sine input

Here we have a signal that is $F(t) = 0$ for $t < 0$ and $F(t) = V_0 sin(\omega_0 t)$ for $t \geq 0$. For focus only on the driven part and take $V(0^-) = 0$.

$$
\begin{aligned}
V(t) &= \int_0^t \frac{dt'}{\tau}\, e^{-(t-t')/\tau}\, V_0 sin(\omega_0 t') \qquad (6.21)\\
&= V_0\, e^{-t/\tau} \int_0^t \frac{dt'}{\tau}\, e^{t'/\tau}\, \frac{e^{i\omega_0 t'} - e^{-i\omega_0 t'}}{2i}\\
&= V_0\, \frac{e^{-t/\tau}}{2i} \int_0^{t/\tau} dx\, e^x \left(e^{i\omega_0 \tau x} - e^{-i\omega_0 \tau x}\right)\\
&= V_0\, \frac{e^{-t/\tau}}{2i} \int_0^{t/\tau} dx\, \left(e^{(1+i\omega_0\tau)x} - e^{(1-i\omega_0\tau)x}\right)\\
&= V_0\, \frac{e^{-t/\tau}}{2i} \left(\frac{e^{t/\tau} e^{i\omega_0 t} - 1}{1 + i\omega_0\tau} - \frac{e^{t/\tau} e^{-i\omega_0 t} - 1}{1 - i\omega_0\tau}\right)\\
&= V_0\, \frac{1}{2i} \left[\left(\frac{e^{i\omega_0 t}}{1 + i\omega_0\tau} - \frac{e^{-i\omega_0 t}}{1 - i\omega_0\tau}\right) - e^{-t/\tau}\left(\frac{1}{1 + i\omega_0\tau} - \frac{1}{1 - i\omega_0\tau}\right)\right]\\
&= V_0\, \frac{1}{2i} \left[\frac{(1 - i\omega_0\tau)e^{i\omega_0 t} - (1 + i\omega_0\tau)e^{-i\omega_0 t} + 2i\omega_0\tau e^{-t/\tau}}{1 + (\omega_0\tau)^2}\right]\\
&= V_0\, \frac{1}{1 + (\omega_0\tau)^2} \left[\frac{e^{i\omega_0 t} - e^{i\omega_0 t}}{2i} - \omega_0\tau \frac{e^{i\omega_0 t} + e^{i\omega_0 t}}{2} + \omega_0\tau e^{-t/\tau}\right]\\
&= V_0 \left[\frac{1}{1 + (\omega_0\tau)^2}\, \sin(\omega_0 \mathrm{t}) - \frac{\omega_0\tau}{1 + (\omega_0\tau)^2}\, \cos(\omega_0 \mathrm{t}) + \frac{\omega_0\tau}{1 + (\omega_0\tau)^2}\, \mathrm{e}^{-\mathrm{t}/\tau}\right]
\end{aligned}
$$

The sine term represents a faithful transmission of the input and the cosine term represents a phase shifted version, while the exponential represents the transient from turning the signal on at t=0. With a bit more algebra, and recalling that $\sin(a - b) = \sin(a)\cos(b)$ - $\cos(a)\sin(b)$, we can simply this.

$$
\begin{aligned}
V(t) &= \frac{V_0}{\sqrt{1 + (\omega_0\tau)^2}} \left[\frac{1}{\sqrt{1 + (\omega_0\tau)^2}} \sin(\omega_0 \mathrm{t}) - \frac{\omega_0\tau}{\sqrt{1 + (\omega_0\tau)^2}} \cos(\omega_0 \mathrm{t}) + \frac{\omega_0\tau}{\sqrt{1 + (\omega_0\tau)^2}} \mathrm{e}^{-\mathrm{t}/\tau}\right]\\
&= \frac{V_0}{\sqrt{1 + (\omega_0\tau)^2}} \left[\sin\Big[\omega_0 \mathrm{t} - \mathrm{atan}(\omega_0\tau)\Big] + \frac{\omega_0\tau}{\sqrt{1 + (\omega_0\tau)^2}} \mathrm{e}^{-\mathrm{t}/\tau}\right] \qquad (6.22)
\end{aligned}
$$

In steady-state, i.e., $t >> \tau$,

$$
V(t) = \frac{V_0}{\sqrt{1 + (\omega_0\tau)^2}} \sin\Big[\omega_0 \mathrm{t} - \mathrm{atan}(\omega_0\tau)\Big] \qquad (6.23)
$$

This same steady-state answer can be derived, simpler, in the frequency domain. This brings up our last issue, the convolution theorem. This relates the convolution integral over time to the product of the Fourier transform of the two variables. i.e., if

$$V_I(t) \;=\; \int_0^t \frac{dt'}{\tau}\, h(t-t')\; F(t'). \tag{6.24}$$

then

$$\begin{aligned} \tilde{V}_I(\omega) &= \mathcal{F}\left(V(t)\right) \\ &= \tilde{h}(\omega)\tilde{F}(\omega). \end{aligned} \tag{6.25}$$

Convolutions in time turn into products in frequency space.